

Unitary-process discrimination with error margin

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We investigate a discrimination scheme between unitary processes. By introducing a margin for the probability of an erroneous guess, this scheme interpolates the two standard discrimination schemes: minimum-error and unambiguous discrimination. We present solutions for two cases. One is the case of two unitary processes with general prior probabilities. The other is the case with a group symmetry: The processes comprise a projective representation of a finite group. In the latter case, we found that unambiguous discrimination is a kind of “all or nothing”: The maximum success probability is either 0 or 1. We also thoroughly analyze how entanglement with an auxiliary system improves discrimination performance.

I. INTRODUCTION

Suppose Alice performs some operation on a quantum system. How can Bob guess what operation Alice has done? Let us assume Bob can prepare the system in any initial state before Alice’s operation. What Bob can do then is to perform some measurement on the system after Alice’s operation, and guess her operation from the outcome of his measurement. This is the problem of discrimination of quantum processes, and it involves two fundamental issues in quantum information theory.

First of all, quantum measurement is statistical in nature, and generally it destroys the state to be measured. It is, therefore, a nontrivial problem for Bob to find the best measurement to discriminate between generally nonorthogonal states after the operation. This is the problem of quantum state discrimination [1–3]. The second issue is entanglement. Bob should prepare the system in the optimal initial state so that his measurement will be most effective. One question is whether and how much the performance of discrimination improves by an input state entangled with an auxiliary system (ancilla). Another problem concerns the usefulness of an entangled input state into the parallel arrangement of processes of the same kind.

The problem of discrimination of processes has received much attention in recent years, and a number of results have been reported. It has been shown that two distinct unitary devices can be perfectly discriminated by a finite number of devices arranged in parallel and an appropriate entangled input state [4, 5]. Discrimination between unitary processes with a group symmetry has also been studied. Owing to the symmetry of the set of processes, one can determine the optimal measurement scheme and discrimination success probability in terms of group representations [6–10]. Thorough analyses of the asymptotic behavior in Lie group estimation can be found in Refs. [11, 12]. Quite recently, a necessary and sufficient condition for when general quantum operations are perfectly discriminated within a finite number of queries was reported [13].

In this paper, we assume Alice’s operations are uni-

tary. With given prior probabilities, she selects one from a finite set of unitary operations. The set of unitary operations and the prior probabilities are known to Bob. For state discrimination, two schemes have been extensively studied: minimum-error discrimination [1] and unambiguous discrimination which allows an inconclusive result [14–17]. Recently, a new scheme has been proposed [18–20], which interpolates the two standard schemes by introducing a margin on the mean probability of an erroneous guess. We adopt this scheme for process discrimination.

We will thoroughly analyze two solvable cases. The first is the case of two unitary processes with arbitrary prior probabilities. Then, we examine the set of processes with group symmetry: The processes comprise a unitary projective representation of a finite group, and prior probabilities are equal. The maximum discrimination success probability is given in terms of dimensions and multiplicities of irreducible representations. We will clarify whether and how much the discrimination performance can be improved by an input state entangled with an ancilla system.

II. UNITARY-PROCESS DISCRIMINATION WITH ERROR MARGIN

Suppose n unitary operations $\{U_i\}_{i=1}^n$ are defined on a quantum system and Alice performs one of the operations U_i with a prior probability η_i . Bob does not know which operation is performed by Alice though he has the knowledge of the set of operations and the prior probabilities. Bob’s task is to optimally guess which operation was performed by Alice. Bob can prepare the quantum system in any state (input state for the process) before Alice’s operation. He can also perform any measurement on the quantum system after Alice’s operation. This is the problem of unitary-process discrimination.

Let us impose an error margin on the mean probability of Bob’s incorrect guess. This is possible by allowing Bob to declare an inconclusive result “I don’t know.” The positive-operator valued measure (POVM) of Bob’s measurement consists of $n + 1$ elements, E_μ ($\mu = 0, \dots, n$),

where measurement outcome $1 \leq \mu \leq n$ means the process is identified with U_μ , and $E_{\mu=0}$ produces the inconclusive result. By P_{U_i, E_μ} we denote the joint probability that the process is U_i ($i = 1, \dots, n$) and the measurement outcome is μ ($\mu = 0, 1, \dots, n$). The probability P_{U_i, E_μ} is given by

$$P_{U_i, E_\mu} = \eta_i \text{tr} U_i \rho U_i^\dagger E_\mu, \quad (1)$$

where ρ is the input state chosen by Bob, which is generally mixed. The success probability of discrimination is then given by

$$P_o \equiv \sum_{i=1}^n P_{U_i, E_i}. \quad (2)$$

The mean probability of error is

$$P_\times \equiv \sum_{i,j=1}^n P_{U_i, E_j} \quad (i \neq j). \quad (3)$$

We impose a margin m on the mean probability of error,

$$P_\times \leq m. \quad (4)$$

Bob's task is to maximize the success probability P_o subject to the constraint $P_\times \leq m$ by choosing the POVM $\{E_\mu\}_{\mu=0}^n$ and the input state ρ in an optimal way.

The input state can generally be mixed. As shown in the following, the optimal input state can be assumed to be a pure state for $m = 1$ and $m = 0$. For a general error margin, we can give a sufficient condition such that the maximum success probability can be attained by a pure input state.

Let us express a general mixed input state by

$$\rho = \sum_a \lambda_a |a\rangle\langle a|, \quad (5)$$

where $\{\lambda_a\}$ is a probability distribution. The discrimination success probability and the margin condition for the mean error probability are given by

$$\begin{aligned} P_o &= \sum_a \lambda_a \left(\sum_i \eta_i \langle a | U_i^\dagger E_i U_i | a \rangle \right) \\ &\equiv \sum_a \lambda_a P(a), \end{aligned} \quad (6)$$

$$\begin{aligned} P_\times &= \sum_a \lambda_a \left(\sum_{i \neq j} \eta_i \langle a | U_i^\dagger E_j U_i | a \rangle \right) \\ &\equiv \sum_a \lambda_a m(a) \leq m. \end{aligned} \quad (7)$$

Here, $P(a)$ and $m(a)$ are the success probability and the error probability, respectively, when the input state is given by $|a\rangle$. Among the pure states $|a\rangle$ in Eq. (5), let $|a_{\max}\rangle$ be the pure state that has the greatest $P(a)$.

When $m = 1$, we can take $|a_{\max}\rangle$ for the input state, since the error-margin condition is inactive in this case. When $m = 0$, we can also take $|a_{\max}\rangle$ for the input state, since Eq. (7) implies all states $|a\rangle$ satisfy the no-error condition, $m(a) = 0$. Thus, the input state can be assumed pure when $m = 1$ or 0 .

For the general error margin, consider the discrimination problem in which the input state is restricted to be pure, and denote the maximum success probability by $P_{\max}^{\text{pure}}(m)$. It is evident that $P_{\max}^{\text{pure}}(m)$ is a monotonically increasing function of m . Note that the inequality $P(a) \leq P_{\max}^{\text{pure}}(m(a))$ holds by definition. Now, assume that $P_{\max}^{\text{pure}}(m)$ is a concave function of m . Then we observe

$$\begin{aligned} P_o &= \sum_a \lambda_a P(a) \leq \sum_a \lambda_a P_{\max}^{\text{pure}}(m(a)) \\ &\leq P_{\max}^{\text{pure}}\left(\sum_a \lambda_a m(a)\right) \\ &\leq P_{\max}^{\text{pure}}(m). \end{aligned}$$

This implies that the success probability of any mixed input state never exceeds the success probability of the optimal pure input state. Thus, the concavity of $P_{\max}^{\text{pure}}(m)$ is a sufficient condition for the maximum success probability to be able to be attained by a pure input state.

This argument holds regardless of the use of an auxiliary system. If we allow a sufficiently large auxiliary system for the input state and measurement, we can show stronger results: $P_{\max}(m) = P_{\max}^{\text{pure}}(m)$ and $P_{\max}(m)$ is concave. Suppose the unitary operations U_i act on system Q and we have an auxiliary system R . Assume the maximum success probability is attained by a generally mixed input state ρ^{QR} and a POVM E_μ^{QR} , which are defined on the composite system QR . Introducing another auxiliary system S , we consider purification of ρ^{QR} , which we denote by $|\Psi^{QRS}\rangle$. If we take $|\Psi^{QRS}\rangle$ as input and measure the output by POVM $E_\mu^{QRS} \equiv E_\mu^{QR} \otimes \mathbf{1}^S$, the success and error probabilities do not change. Thus, the optimality can always be achieved by a pure-state input if a sufficiently large ancilla can be used.

The concavity of $P_{\max}(m)$ can be shown in the following way: Take two values of error margin, m_1 and m_2 . For each error margin m_j ($j = 1, 2$), we assume the set of input state ρ_j^{QR} and POVM $E_\mu^{QR}(j)$ is optimal. Here, we introduce a two-dimensional auxiliary system S with an orthonormal basis $|1^S\rangle$ and $|2^S\rangle$. In the composite system QRS , let us consider the input state

$$\rho^{QRS} = \lambda_1 \rho_1^{QR} \otimes |1^S\rangle\langle 1^S| + \lambda_2 \rho_2^{QR} \otimes |2^S\rangle\langle 2^S|,$$

where $\lambda_1, \lambda_2 \geq 0$ and $\lambda_1 + \lambda_2 = 1$, and POVM defined by

$$E_\mu^{QRS} = E_\mu^{QR}(1) \otimes |1^S\rangle\langle 1^S| + E_\mu^{QR}(2) \otimes |2^S\rangle\langle 2^S|.$$

It is clear that the mean error probability is given by $P_\times = \lambda_1 m_1 + \lambda_2 m_2$. The mean success probability is also given by a similar form, $P_o = \lambda_1 P_{\max}(m_1) + \lambda_2 P_{\max}(m_2)$, which should not exceed the maximum success probability $P_{\max}(\lambda_1 m_1 + \lambda_2 m_2)$. This establishes the concavity of $P_{\max}(m)$ when a sufficiently large ancilla is available.

III. DISCRIMINATION BETWEEN TWO UNITARY PROCESSES

In this section, we consider discrimination with error margin between two unitary processes U_1 and U_2 with prior probabilities η_1 and η_2 , respectively.

First, we assume the input state is fixed to be a certain pure state $|\phi\rangle$. Optimization is performed only with respect to POVM. Then, the problem reduces to discrimination between two pure states $|\phi_1\rangle \equiv U_1|\phi\rangle$ and $|\phi_2\rangle \equiv U_2|\phi\rangle$. This problem has already been solved in [19, 20]. One of the three types of measurement is optimal depending on the following parameters: prior probabilities η_i , the inner product between the two states, and error margin m . The parameter space is divided into the following three domains:

$$\begin{cases} \text{minimum-error domain :} & m_c \leq m \leq 1, \\ \text{intermediate domain :} & m'_c \leq m \leq m_c, \\ \text{single-state domain :} & 0 \leq m \leq m'_c, \end{cases}$$

where two critical error margins m_c and m'_c are defined by

$$m_c \equiv \frac{1}{2} \left(1 - \sqrt{1 - 4\eta_1\eta_2 S} \right), \quad (8)$$

$$m'_c \equiv \begin{cases} \frac{(\eta_1 - \sqrt{\eta_1\eta_2 S})^2}{1 - 2\sqrt{\eta_1\eta_2 S}} & (\eta_1 \leq \eta_2 S), \\ 0 & (\eta_1 \geq \eta_2 S). \end{cases} \quad (9)$$

where $\eta_1 \leq \eta_2$ is assumed and $S \equiv |\langle \phi_1 | \phi_2 \rangle|^2$. In the minimum-error domain, the optimal measurement is the same as the one of minimum-error discrimination, which does not produce the inconclusive result, “I don’t know.” In the single-state domain, one of the two states is omitted in the optimal measurement. In the intermediate domain, the probabilities for three measurement outcomes (the states $|\phi_1\rangle$ and $|\phi_2\rangle$ and the inconclusive result) are nonzero. The maximum success probability as a function of m and S is given by

$$P_{\max}^{\text{pure}}(m, S) = \begin{cases} \frac{1}{2} (1 + \sqrt{1 - 4\eta_1\eta_2 S}) & (m_c \leq m \leq 1), \\ \left(\sqrt{m} + \sqrt{1 - 2\sqrt{\eta_1\eta_2 S}} \right)^2 & (m'_c \leq m \leq m_c), \\ \eta_2 \left(\sqrt{\frac{m}{\eta_1}} S + \sqrt{\frac{\eta_1 - m}{\eta_1}} (1 - S) \right)^2 & (0 \leq m \leq m'_c). \end{cases} \quad (10)$$

See Refs. [19, 20] for details. It can be readily shown that $P_{\max}^{\text{pure}}(m, S)$ is a concave and monotonically increasing function of m . It is also evident that $P_{\max}^{\text{pure}}(m, S)$ is monotonically decreasing as a function of S .

We can now optimize the success probability with respect to the input pure state $|\phi\rangle$ in the following way:

$$\begin{aligned} P_{\max}^{\text{pure}}(m) &= \max_{|\phi\rangle} P_{\max}^{\text{pure}}(m, |\langle \phi | U_1^\dagger U_2 | \phi \rangle|^2) \\ &= P_{\max}^{\text{pure}}(m, S_{\min}), \end{aligned} \quad (11)$$

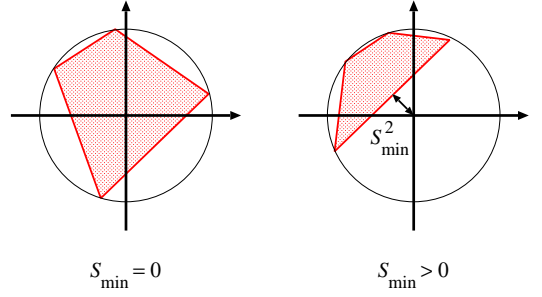


FIG. 1: (Color online) S_{\min} and the convex polygon on the complex plane which is the convex hull of the eigenvalues of $U_1^\dagger U_2$.

where S_{\min} is defined to be

$$S_{\min} \equiv \min_{|\phi\rangle} \left| \langle \phi | U_1^\dagger U_2 | \phi \rangle \right|^2. \quad (12)$$

Since $P_{\max}^{\text{pure}}(m) = P_{\max}^{\text{pure}}(m, S_{\min})$ is concave for m , we conclude that the maximum success probability of discriminating two unitary processes can be attained by a pure state input; namely, $P_{\max}(m) = P_{\max}^{\text{pure}}(m)$.

S_{\min} can be determined by eigenvalues of $U_1^\dagger U_2$ [4, 5]. Let $\{e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_d}\}$ be eigenvalues of $U_1^\dagger U_2$, where d is the dimension of the space considered. We can express S_{\min} as

$$\begin{aligned} S_{\min} &= \min_{|\phi\rangle} \left| \sum_{a=1}^d |\langle \phi | a \rangle|^2 e^{i\theta_a} \right|^2 \\ &= \min_{q_a \geq 0, \sum_a q_a = 1} \left| \sum_{a=1}^d q_a e^{i\theta_a} \right|^2, \end{aligned} \quad (13)$$

where $|a\rangle$ is the eigenstate of $U_1^\dagger U_2$ with eigenvalue $e^{i\theta_a}$, and q_a is defined to be $|\langle \phi | a \rangle|^2$. We note that $\sum_{a=1}^d q_a e^{i\theta_a}$ with $q_a \geq 0, \sum_a q_a = 1$ is the convex hull of the points $\{e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_d}\}$ on the complex plane, and represents a convex polygon on the plane. If this polygon contains the origin, then $S_{\min} = 0$, and consequently we obtain $P_{\max}(m) = 1$. Otherwise, S_{\min} is given by the square of the minimum distance between the polygon and the origin (see Fig. 1).

Before concluding the section, we examine whether some use of an auxiliary system (ancilla) can help Bob improve the discrimination success probability. Suppose Alice’s two unitary operations act on the system Q alone, and Bob can prepare a certain input state in the composite system QR and perform any measurement on QR after Alice’s operation, where R is an auxiliary system. The set of eigenvalues of the operator $U_1^\dagger U_2$ is unchanged with only their multiplicities increased by a factor of the dimension of system R . This does not change S_{\min} . Thus, entanglement with an auxiliary system does not help in discrimination between two unitary processes. This contrasts with the case of discrimination discussed in the

next section, which involves more than two unitary processes.

IV. UNITARY PROCESSES AS A PROJECTIVE REPRESENTATION OF A FINITE GROUP

It is generally hard to analyze a process discrimination problem of more than two processes. However, if the set of processes has some symmetry, the problem can be tractable. In this section, we consider a set of processes with a group symmetry. There are many interesting cases where a set of operations itself forms a group. Here, we consider a slightly generalized situation: The set of unitary processes $\{U_g\}_{g \in G}$ is assumed to be a unitary projective representation of a finite group G .

More precisely, the set of unitary processes $\{U_g\}_{g \in G}$ satisfies

$$U_g U_h = c_{g,h} U_{gh} \quad (g, h \in G), \quad (14)$$

where $c_{g,h}$ is a complex number with $|c_{g,h}| = 1$. The occurrence probabilities of process U_g are assumed to be equal: $\eta_g = \frac{1}{|G|}$ with $|G|$ being the order of G .

When all $c_{g,h}$ are 1, the projective representation reduces to an ordinary unitary representation of G . The set of complex numbers $\{c_{g,h}\}$ is called a factor set and satisfies the following cocycle conditions:

$$c_{g,h} c_{gh,k} = c_{g,hk} c_{h,k} \quad (g, h, k \in G), \quad (15)$$

which is a consequence of the associativity of multiplication, $U_g U_h U_k$. The factor set depends on phase factors of each U_g . It is important that the factor set does not always reduce to a trivial one (all $c_{g,h}$ are 1) by redefining phase factors of U_g . A simple example is the set $\{\mathbf{1}, \sigma_x, \sigma_y, \sigma_z\}$, where σ_x, σ_y , and σ_z are Pauli matrices. This set is not a group, but a projective representation of group $\mathbb{Z}_2 \times \mathbb{Z}_2$. We will later discuss a generalization of this example in connection with superdense coding.

For a general theory of projective representation of group (also known as ray representation) see, for example, Ref. [21]. It is known that, for a fixed factor set, equivalence, reducibility, and irreducibility can be defined in the same way as in ordinary representations. Schur's lemma and the orthogonality relations of irreducible representation matrices also hold for projective representations.

Further, in most of our calculations involving U_g , the factor set does not explicitly show up. For example, the relation $U_g U_h A U_h^\dagger U_g^\dagger = U_{gh} A U_{gh}^\dagger$ still holds for any operator A . We also observe $U_1 A U_1^\dagger = A$ and $U_g A U_g^\dagger = U_{g^{-1}}^\dagger A U_{g^{-1}}$, where "1" represents the identity element of G . This is because, up to a phase factor, $U_1 = \mathbf{1}$ and $U_g^\dagger = U_{g^{-1}}$.

A POVM $\{E_0, E_g\}$ is said to be covariant [2], if it satisfies

$$U_g E_0 U_g^\dagger = E_0, \quad U_h E_g U_h^\dagger = E_{hg} \quad (g, h \in G).$$

We can show that the optimal POVM can be assumed to be covariant. This useful property is not hampered by the error-margin condition and the factor set in the projective representation. Let $\{F_0, F_g\}$ be a POVM, which is not generally covariant. Construct another POVM $\{E_0, E_g\}$ as follows:

$$E_0 = \frac{1}{|G|} \sum_{g \in G} U_g F_0 U_g^\dagger, \quad E_g = U_g E_1 U_g^\dagger, \quad (16)$$

where $E_1 \equiv \frac{1}{|G|} \sum_{g \in G} U_g^\dagger F_g U_g$. It is evident that $\{E_0, E_g\}$ is a covariant POVM. We find that the two POVMs give the same success probability.

$$\begin{aligned} P_o(F) &\equiv \frac{1}{|G|} \sum_{g \in G} \text{tr} F_g U_g \rho U_g^\dagger = \text{tr} E_1 \rho \\ &= \frac{1}{|G|} \sum_{g \in G} \text{tr} E_g U_g \rho U_g^\dagger \equiv P_o(E). \end{aligned}$$

The error probability is also the same for the two POVMs.

$$\begin{aligned} P_\times(F) &\equiv \frac{1}{|G|} \sum_{g \neq h} \text{tr} F_g U_h \rho U_h^\dagger = \sum_{g(\neq 1)} \text{tr} E_1 U_g \rho U_g^\dagger \\ &= \frac{1}{|G|} \sum_{g \neq h} \text{tr} E_g U_h \rho U_h^\dagger \equiv P_\times(E). \end{aligned}$$

Thus, if a POVM $\{F_0, F_g\}$ is optimal, so is the covariant POVM $\{E_0, E_g\}$ which is constructed from $\{F_0, F_g\}$.

In terms of the covariant POVM, the task is to maximize

$$P_o = \text{tr} E_1 \rho, \quad (17)$$

subject to conditions

$$E_1 \geq 0, \quad \sum_{g \in G} U_g E_1 U_g^\dagger \leq \mathbf{1}, \quad (18)$$

$$P_\times = \sum_{g \neq 1} \text{tr} E_1 U_g \rho U_g^\dagger \leq m, \quad (19)$$

where variables are POVM element E_1 and input state ρ .

A. Case of irreducible representation

Let us assume the representation U_g is irreducible and determine the maximum success probability $P_{\max}(m)$. We will see that this simple case provides a helpful guide line for the more general case considered in the next section.

Define an operator A to be $\sum_{g \in G} U_g E_1 U_g^\dagger$. It can be readily shown that A commutes with U_g for all g in G . According to Schur's lemma, operator A is the identity

up to a factor, since the representation U_g is irreducible. The factor can be fixed by calculating traces. We find

$$\sum_{g \in G} U_g E_1 U_g^\dagger = \frac{|G| \text{tr } E_1}{d} \mathbf{1}, \quad (20)$$

where d is the dimension of the space considered. The POVM element E_1 is positive semidefinite, and it clearly satisfies $\text{tr}(E_1) \mathbf{1} \geq E_1$. Combining this inequality and Eq. (20), we obtain

$$E_1 \leq \frac{d}{|G|} \sum_{g \in G} U_g E_1 U_g^\dagger, \quad (21)$$

which serves as a key inequality for determining $P_{\max}(m)$.

Using this inequality, we derive two upper bounds for the success probability P_o . The first upper bound is obtained in the following way:

$$P_o = \text{tr } E_1 \rho \leq \frac{d}{|G|} \text{tr} \left(\sum_{g \in G} U_g E_1 U_g^\dagger \rho \right) \leq \frac{d}{|G|}, \quad (22)$$

where the condition

$$\sum_{g \in G} U_g E_1 U_g^\dagger \leq \mathbf{1},$$

is used in the last inequality. This upper bound is independent of the error margin. To obtain another upper bound involving the error margin, we slightly rewrite the inequality of Eq. (21) as

$$E_1 \leq \frac{d}{|G|} \left(E_1 + \sum_{g \neq 1} U_g E_1 U_g^\dagger \right),$$

which immediately leads to

$$E_1 \leq \frac{1}{\frac{|G|}{d} - 1} \sum_{g \neq 1} U_g E_1 U_g^\dagger. \quad (23)$$

We note that $|G| \geq d$ for an irreducible representation and equality occurs only for a trivial case $|G| = 1$. Hereafter, we assume $|G| > 1$. Using this inequality, we obtain the second upper bound as

$$\begin{aligned} P_o = \text{tr } E_1 \rho &\leq \frac{1}{\frac{|G|}{d} - 1} \text{tr} \left(\sum_{g \neq 1} U_g E_1 U_g^\dagger \rho \right) \\ &= \frac{1}{\frac{|G|}{d} - 1} P_\times \leq \frac{m}{\frac{|G|}{d} - 1}. \end{aligned} \quad (24)$$

Combining the two upper bounds, we have

$$\begin{aligned} P_o &\leq \begin{cases} \frac{d}{|G|} & \left(1 - \frac{d}{|G|} \leq m \leq 1 \right), \\ \frac{m}{\frac{|G|}{d} - 1} & \left(0 \leq m < 1 - \frac{d}{|G|} \right), \end{cases} \\ &\equiv f(m). \end{aligned} \quad (25)$$

It is readily verified that this upper bound can be attained by any pure state input $\rho = |\phi\rangle\langle\phi|$ and the POVM element $E_1 = f(m)|\phi\rangle\langle\phi|$. Thus, the maximum success probability in the case of an irreducible representation is given by

$$P_{\max}(m) = P_{\max}^{\text{pure}}(m) = \begin{cases} \frac{d}{|G|} & (m_c \leq m \leq 1), \\ \frac{m}{\frac{|G|}{d} - 1} & (0 \leq m < m_c), \end{cases} \quad (26)$$

where the critical error margin m_c is defined as

$$m_c = 1 - \frac{d}{|G|}. \quad (27)$$

If $m \geq m_c$, $P_{\max}(m)$ is given by that of minimum-error discrimination. Below m_c , $P_{\max}(m)$ is linear in error margin m . Interestingly, we find that $P_{\max}(0) = 0$ unless $|G| = d$, implying it is impossible to unambiguously discriminate processes U_g unless $|G| = d$. If $|G| = d$ instead, we find $P_{\max}(0) = 1$. Thus, $P_{\max}(0)$ is either 0 or 1. Rather surprisingly, we will see that these features of the irreducible case are preserved in the more general case discussed next.

B. General case

Here, we consider the representation U_g to be generally reducible. When U_g is irreducible, the inequality (21) was essential to determination of the maximum success probability. Though we cannot resort to Schur's lemma as in the preceding section, there exists a generalization of Eq. (21) for generally reducible representations. In this respect, we can show that the following general theorem holds:

Theorem. *Let $\{U_g\}_{g \in G}$ be a unitary projective representation of a finite group G of order $|G|$. Define constant κ as*

$$\kappa \equiv \sum_{\sigma} \frac{\min(m_{\sigma}, d_{\sigma}) d_{\sigma}}{|G|}, \quad (28)$$

where σ represents each irreducible representation of G , and d_{σ} and m_{σ} are the dimension and the multiplicity of irreducible representation σ in the decomposition of U_g , respectively. Then, for any positive semidefinite operator E , the following inequality holds:

$$E \leq \kappa \sum_{g \in G} U_g E U_g^\dagger. \quad (29)$$

The quantity $d_{\sigma}^2/|G|$ is called the Plancherel measure of irreducible representation σ . It is known that they sum to unity when summed over all possible irreducible representations for a given factor set [21]. Thus, the constant κ is generally less than or equal to 1. Note that if U_g is irreducible, the generalized inequality Eq. (29) reduces to Eq. (21), since κ is then given by $d/|G|$.

Before proving the theorem, we introduce a representation basis and representation matrices. Decomposing the representation U_g into irreducible representations, we obtain an orthonormal basis written as $|\sigma, b, a\rangle$ ($a = 1, \dots, d_\sigma, b = 1, \dots, m_\sigma$). Here, σ represents each irreducible representation of G . Index a specifies each vector belonging to irreducible representation σ , and a runs from 1 to d_σ . Index b stands for “other quantum numbers,” which are invariant under any operation U_g . Index b , therefore, runs from 1 to the multiplicity m_σ . The basis states $|\sigma, b, a\rangle$ transform under operation U_g as follows:

$$\begin{aligned} U_g |\sigma, b, a\rangle &= \sum_{a'=1}^{d_\sigma} |\sigma, b, a'\rangle \langle \sigma, b, a' | U_g | \sigma, b, a \rangle \\ &= \sum_{a'=1}^{d_\sigma} D_{a'a}^\sigma(g) |\sigma, b, a'\rangle. \end{aligned} \quad (30)$$

Here, $D_{a'a}^\sigma(g) \equiv \langle \sigma, b, a' | U_g | \sigma, b, a \rangle$ are irreducible representation matrices, and that are known to satisfy the following orthogonal relations:

$$\sum_{g \in G} D_{a_1 a_2}^{\sigma*}(g) D_{a'_1 a'_2}^\sigma(g) = \delta_{\sigma\sigma'} \delta_{a_1 a'_1} \delta_{a_2 a'_2} \frac{|G|}{d_\sigma}. \quad (31)$$

We now present the proof of the theorem.

Proof. Any positive semidefinite operator E can be written as

$$E = \sum_e |e\rangle \langle e|, \quad (32)$$

where $|e\rangle$ is not generally normalized. If each term in this expression satisfies inequality (29), so does E . Thus, it suffices to prove the case in which the rank of E is one: $E = |e\rangle \langle e|$. The vector $|e\rangle$ can be expanded in terms of the basis $|\sigma, b, a\rangle$ as

$$|e\rangle = \sum_\sigma \sum_{b=1}^{m_\sigma} \sum_{a=1}^{d_\sigma} e_{\sigma ba} |\sigma, b, a\rangle.$$

Here, it is convenient to redefine the basis $|\sigma, b, a\rangle$ so that the coefficient $e_{\sigma ba}$ is diagonal with respect to b and a . This is possible by the singular value decomposition of the matrix with (b, a) entry given by $e_{\sigma ba}$. Note that the transformation property of Eq. (30) is unchanged by this redefinition. In this redefined basis, we can write

$$|e\rangle = \sum_\sigma \sum_{a=1}^{\tilde{d}_\sigma} e_{\sigma a} |\sigma, a, a\rangle, \quad (33)$$

where $\tilde{d}_\sigma \equiv \min(m_\sigma, d_\sigma)$. Using the orthogonality of irreducible matrices given in Eq. (31), we obtain

$$\begin{aligned} \sum_{g \in G} U_g E U_g^\dagger &= \sum_{g \in G} U_g |e\rangle \langle e| U_g^\dagger \\ &= \sum_\sigma \sum_{a=1}^{\tilde{d}_\sigma} \frac{|G|}{d_\sigma} |e_{\sigma a}|^2 \sum_{a'=1}^{d_\sigma} |\sigma, a, a'\rangle \langle \sigma, a, a'|. \end{aligned} \quad (34)$$

Now, let $|\phi\rangle$ be an arbitrary state. Writing $\phi_{\sigma ba} \equiv \langle \sigma, b, a | \phi \rangle$, we find

$$\begin{aligned} \langle \phi | E | \phi \rangle &= |\langle \phi | e \rangle|^2 = \left| \sum_\sigma \sum_{a=1}^{\tilde{d}_\sigma} e_{\sigma a} \phi_{\sigma aa}^* \right|^2 \\ &= \left| \sum_\sigma \sum_{a=1}^{\tilde{d}_\sigma} \sqrt{\frac{d_\sigma}{|G|}} \sqrt{\frac{|G|}{d_\sigma}} e_{\sigma a} \phi_{\sigma aa}^* \right|^2 \\ &\leq \left(\sum_\sigma \sum_{a=1}^{\tilde{d}_\sigma} \frac{d_\sigma}{|G|} \right) \left(\sum_\sigma \sum_{a=1}^{\tilde{d}_\sigma} \frac{|G|}{d_\sigma} |e_{\sigma a}|^2 |\phi_{\sigma aa}|^2 \right), \end{aligned}$$

where the Schwarz inequality is used. The first factor in the last line is the constant κ defined by Eq. (28). To evaluate the second factor, we use Eq. (34) and obtain

$$\langle \phi | \sum_{g \in G} U_g E U_g^\dagger | \phi \rangle = \sum_\sigma \sum_{a=1}^{\tilde{d}_\sigma} \frac{|G|}{d_\sigma} |e_{\sigma a}|^2 \sum_{a'=1}^{d_\sigma} |\phi_{\sigma aa'}|^2,$$

which is clearly greater than or equal to the second factor. Thus, we obtain

$$\langle \phi | E | \phi \rangle \leq \kappa \langle \phi | \sum_{g \in G} U_g E U_g^\dagger | \phi \rangle.$$

Equality holds if and only if $e_{\sigma a}^* \phi_{\sigma aa} / d_\sigma$ is independent of σ and a , and $\phi_{\sigma ba} = 0$ for $b \neq a$. Since $|\phi\rangle$ is arbitrary, we obtain the desired result. This completes the proof of the theorem. \square

With the key inequality of Eq. (29) at hand, we can determine the maximum success probability along the same lines as the irreducible case. The key inequality immediately leads to the first upper bound for the success probability

$$P_o(m) \leq \kappa. \quad (35)$$

By rewriting the key inequality as in the irreducible case, we have

$$E_1 \leq \frac{\kappa}{1-\kappa} \sum_{g \neq 1} U_g E_1 U_g^\dagger, \quad (36)$$

which is a generalization of Eq. (23). The second upper bound follows from this inequality.

$$P_o \leq \frac{\kappa}{1-\kappa} m. \quad (37)$$

Combining the two upper bounds, we have

$$\begin{aligned} P_o &\leq \begin{cases} \kappa & (1-\kappa \leq m \leq 1) \\ \frac{\kappa}{1-\kappa} m & (0 \leq m < 1-\kappa), \end{cases} \\ &\equiv f(m). \end{aligned} \quad (38)$$

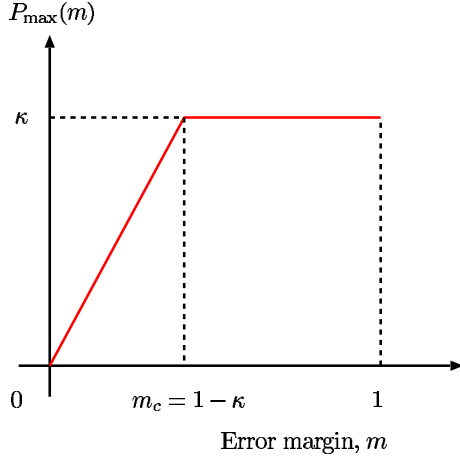


FIG. 2: (Color online) The maximum discrimination success probability $P_{\max}(m)$ of processes $\{U_g\}_{g \in G}$ which comprise a projective representation of a finite group G . m denotes the margin for the mean error probability. The constant κ is given in Eq. (28), which is replaced by κ^A in Eq. (44) when a sufficiently large ancilla system can be employed.

This upper bound is attained by the following pure-state input:

$$|\phi\rangle = \frac{1}{\sqrt{\kappa}} \sum_{\sigma} \sum_{a=1}^{\tilde{d}_{\sigma}} \sqrt{\frac{d_{\sigma}}{|G|}} |\sigma, a, a\rangle, \quad (39)$$

and the POVM element E_1 of rank 1 given by

$$E_1 = f(m) |\phi\rangle\langle\phi|. \quad (40)$$

Thus, the maximum success probability $P_{\max}(m)$ is given by

$$P_{\max}(m) = P_{\max}^{\text{pure}}(m) = \begin{cases} \kappa & (m_c \leq m \leq 1), \\ \frac{\kappa}{1-\kappa} m & (0 \leq m < m_c), \end{cases} \quad (41)$$

where $m_c = 1 - \kappa$ and $\kappa = \sum_{\sigma} \frac{\min(m_{\sigma}, d_{\sigma}) d_{\sigma}}{|G|}$. For minimum-error discrimination ($m = 1$), this maximum success probability reproduces the result obtained in [9]. As in the irreducible case, we find again that $P_{\max}(m)$ is linear below the critical error margin m_c and reaches the probability of minimum-error discrimination at $m = m_c$ (see Fig. 2). Unambiguous discrimination is again “all or nothing”: $P_{\max}(0)$ is either 0 or 1. These features contrast with the case of discrimination between two unitary processes with no group symmetry. Note that a set of two unitaries cannot always be considered as a projective representation of some group. This is because the group should be \mathbb{Z}_2 in this case, and this would mean, up to a phase factor, the square of $U_1^\dagger U_2$ should be the identity.

V. ENTANGLEMENT WITH ANCILLA AND ILLUSTRATIVE EXAMPLES

We examine how entanglement with an ancilla system improves the discrimination performance. Let Q be the system on which the process U_g^Q acts, and let R be an ancilla system. We assume that the input state can be any (generally entangled) state $|\phi\rangle^{QR}$ of the composite system QR , and any measurement on system QR can be performed after the operation of U_g^Q .

Clearly, $U_g^Q \otimes \mathbf{1}^R$ is a projective representation of G , and the arguments given in the preceding section can be applied to the composite system QR as well. We write the basis of irreducible representations in QR as

$$|\sigma, (b, r), a\rangle^{QR} \equiv |\sigma, b, a\rangle^Q \otimes |r\rangle^R, \quad (42)$$

where $\{|r\rangle\}$ is an orthonormal basis of R . By introducing the ancilla, the multiplicity of irreducible representation σ is increased by a factor of the dimension of the space of R , which we denote by $|R|$. The maximum success probability is still given by Eq. (41), but with the constant κ replaced by

$$\kappa' = \sum_{\sigma} \frac{\min(m_{\sigma} |R|, d_{\sigma}) d_{\sigma}}{|G|}, \quad (43)$$

and, for an ancilla of a sufficiently large dimension, by

$$\kappa^A = \sum_{\sigma(m_{\sigma} \geq 1)} \frac{d_{\sigma}^2}{|G|}, \quad (44)$$

which is the Plancherel measure of irreducible representations appearing in the representation U_g^Q [10]. Note that $\kappa^A = 1$ if all irreducible representations are in the decomposition of U_g . Since $\kappa \leq \kappa' \leq \kappa^A$, the success probability generally improves by an ancilla. It should be noted that any ancilla is useless ($\kappa = \kappa^A$) if all irreducible representations are one dimensional. This applies to an ordinary representation of an Abelian group. For a nontrivial projective representation of an Abelian group, however, this is not always true, which will be illustrated by an example later.

Let us focus on the unambiguous discrimination case ($m = 0$). Without an ancilla, the success probability is 1 when $m_{\sigma} \geq d_{\sigma}$ for all possible irreducible representations σ . Otherwise, it is 0. If an irreducible representation σ is missing in the decomposition of U_g^Q , meaning $m_{\sigma} = 0$, any ancilla does not help. This is because the missing representation does not appear with any ancilla. The most interesting case is probably when no irreducible representation is missing ($m_{\sigma} \geq 1$ for all σ), but $m_{\sigma} < d_{\sigma}$ for some σ . Then, the success probability without an ancilla is 0. With a sufficiently large ancilla, however, the maximum success probability becomes 1.

In what follows, we present three examples, which illustrate some differences in usefulness of an ancilla system.

A. Phase shift discrimination

Consider the following phase shift processes on a qubit:

$$\begin{cases} U_k|0\rangle = |0\rangle, \\ U_k|1\rangle = e^{i\frac{2\pi}{K}k}|1\rangle, \end{cases} \quad k = 0, 1, \dots, K-1, \quad (45)$$

where K is a positive integer. $\{U_k\}_{k=0}^{K-1}$ is an ordinary representation of the Abelian group \mathbb{Z}_K . All irreducible representations of \mathbb{Z}_K are one dimensional and specified by an integer σ ($= 0, 1, \dots, K-1$) as $D^\sigma(k) = e^{i\frac{2\pi}{K}\sigma k}$. The representation U_k contains two irreducible representations, $\sigma = 0$ and $\sigma = 1$. As mentioned, an ancilla is useless for any error margin in this example. We find that the maximum success probability is given by Eq. (41) with $\kappa = \kappa^A = 2/K$.

This example provides one of the cases in which we can easily calculate the maximum success probability when the operations are performed on N identical systems in parallel, and the input state of the N systems is allowed to be entangled among its subsystems. The processes are now expressed as $U_k^{\otimes N}$ for an N -qubit system. We observe

$$U_k^{\otimes N}|b_1 b_2 \dots b_N\rangle = e^{i\frac{2\pi}{K}k(b_1 + b_2 + \dots + b_N)}|b_1 b_2 \dots b_N\rangle,$$

which shows that each computational basis state of the N -qubit system is an irreducible representation, with σ given by the number of entries of 1. Thus, for $N \leq K-2$, we find $\kappa = (N+1)/K$, and for $N \geq K-1$, we find $\kappa = 1$.

B. Quantum color coding

The second example is quantum color coding [9, 10]. Consider N identical quantum systems, each defined on vector space \mathbb{C}^d . Suppose Alice randomly permutes the N quantum systems. Bob's task is to identify which permutation was performed by Alice. The dimension d can be interpreted as the number of colors, and N as the number of colored boxes to be identified. Alice's operations comprise a set of $N!$ permutation processes U_g on $(\mathbb{C}^d)^{\otimes N}$, which is an ordinary representation of the symmetric group of degree N .

If $d \geq N$, it is clear that Bob can discriminate Alice's permutation with certainty. If $d < N$, κ and κ^A are less than 1. For small N , differences between κ and κ^A are not remarkable. For example, when $N = 4$ and $d = 2$, we find $\kappa = 1/2$ and $\kappa^A = 7/12$. However, in the large- N limit, κ goes to 1 if $d > N/e$ [9], and κ^A goes to 1 if $d > 2\sqrt{N}$ [10]. Thus, for $m > 0$, entanglement with an ancilla improves the discrimination performance substantially (see Fig. 3).

For unambiguous discrimination ($m = 0$), however, ancilla does not help. If $d < N$, some irreducible representations, for example, the totally antisymmetric representation, are missing in the decomposition of U_g . Then, with any ancilla, Bob cannot unambiguously discriminate Alice's permutation.

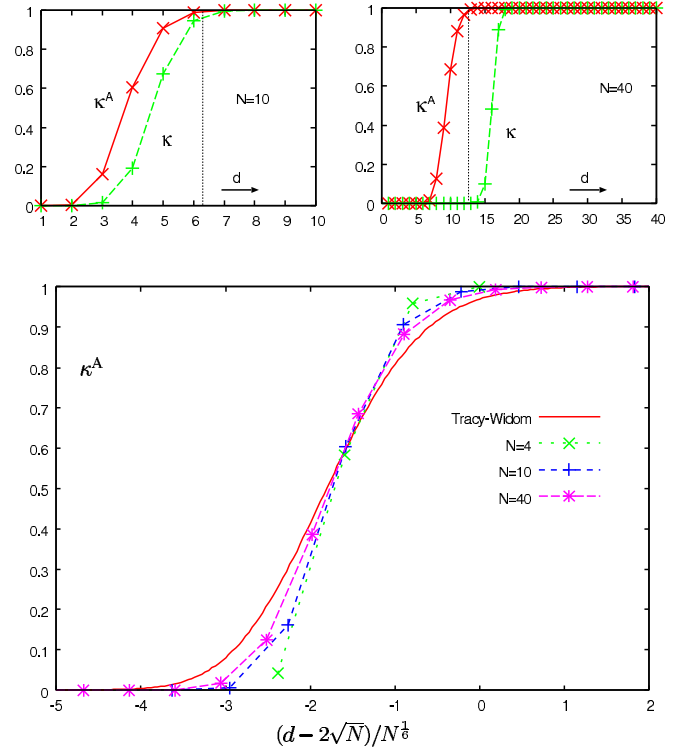


FIG. 3: (Color online) The constants κ^A and κ in the quantum color coding, which are the maximum probabilities in minimum-error discrimination with and without an ancilla system, respectively. N is the number of colored boxes, and d is the number of colors. In the upper two figures, κ and κ^A are compared. The vertical dotted lines indicate the positions of $d = 2\sqrt{N}$. In the bottom figure, κ^A are plotted as functions of $(d - 2\sqrt{N})/N^{1/6}$, so that they approach the Tracy-Widom distribution in the large- N limit [10].

C. Superdense coding

The last example is the celebrated superdense coding in general dimensions [22]. Consider the following unitary operators on \mathbb{C}^d :

$$U_{k,l} = X^k Z^l \quad (k, l = 0, \dots, d-1), \quad (46)$$

where X and Z are generalizations of Pauli matrices σ_x and σ_z , respectively, given by

$$\begin{aligned} X &= \sum_{a=0}^{d-1} |a\rangle\langle a+1|, \\ Z &= \sum_{a=0}^{d-1} e^{i\frac{2\pi}{d}a} |a\rangle\langle a|. \end{aligned}$$

By the relation $XZ = e^{i\frac{2\pi}{d}}ZX$, we have

$$U_{k,l}U_{k',l'} = e^{-i\frac{2\pi}{d}lk'}U_{k+k',l+l'},$$

which shows that $U_{k,l}$ is a projective representation of $\mathbb{Z}_d \times \mathbb{Z}_d$. The factor set is given by $c_{(k,l),(k',l')} = e^{-i\frac{2\pi}{d}lk'}$.

Remember that equivalence of projective representations is defined for a fixed factor set. For this factor set, $U_{k,l}$ turns out to be the unique irreducible representation of $\mathbb{Z}_d \times \mathbb{Z}_d$. The dimension of this unique irreducible representation is d and its multiplicity is 1, which gives $\kappa = \frac{1}{d}$ and $\kappa^A = 1$. Thus, without ancilla, the unambiguous discrimination probability is 0. However, we can perfectly discriminate the d^2 processes $U_{k,l}$ by using an ancilla system of dimension d . In fact, as is well known, the states $|\phi_{k,l}\rangle^{QR} = U_{k,l}^Q \otimes \mathbf{1}^R |\phi\rangle^{QR}$ are mutually orthogonal if we take the following entangled input state:

$$|\phi\rangle^{QR} = \frac{1}{\sqrt{d}} \sum_{a=0}^{d-1} |a\rangle^Q \otimes |a\rangle^R.$$

This example clearly shows the difference between ordinary representations and nontrivial projective representations of the same group. Consider the following phase shift operations for a qutrit system:

$$V_{k,l} = \text{diag} \left(1, e^{i\frac{2\pi}{d}k}, e^{i\frac{2\pi}{d}l} \right) \quad (k, l = 0, \dots, d-1), \quad (47)$$

which is an ordinary representation of the Abelian group $\mathbb{Z}_d \times \mathbb{Z}_d$. Any ancilla is useless for $V_{k,l}$, though $V_{k,l}$ and $U_{k,l}$ are both representations of the same group $\mathbb{Z}_d \times \mathbb{Z}_d$.

VI. CONCLUDING REMARKS

We have studied unitary-process discrimination with an error margin. By imposing a margin on the mean error

probability, this scheme interpolates minimum-error and unambiguous discrimination.

Two cases have been thoroughly analyzed and solutions were presented. One is the case of two unitary processes with arbitrary prior probabilities. The other is the set of processes with group symmetry: The processes comprise a unitary projective representation of a finite group, and prior probabilities are equal. Especially, in the latter case, we clarified the conditions under which discrimination performance improves by an input state entangled with an ancilla system. This analysis is quite general and applicable to many interesting unitary-process discrimination problems with a group symmetry. It will also be of interest in the future studies to extend our scheme to discrimination problems of isometries, some classes of a group, and quantum channels with group symmetry.

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